Assignment 3

Hand in no. 1, 4, 5 and 6 by September 26 .

1. A bounded function f on $[a, b]$ is said to be locally Lipschitz continuous at $x \in [a, b]$ if there exist some L and δ such that
$$|f(y) - f(x)| \le L|x - y| \ , \quad \forall y \in (x - \delta, x + \delta).$$
Show that f is Lipschitz continuous at x.

2. Let f be a function defined on (a, b) and $x_0 \in (a, b)$.
   (a) Show that f is Lipschitz continuous at x_0 if its left and right derivatives exist at x_0.
   (b) Construct a function Lipschitz continuous at x_0 whose one sided derivatives do not exist.

3. Can you find a cosine series which converges uniformly to the sine function on $[0, \pi]$? If yes, find one.

4. A sequence $\{a_n\}, n \ge 0$, is said to converge to a in mean if
$$\frac{a_0 + a_1 + \cdots + a_n}{n+1} \to a \ , \quad n \to \infty \ .$$
   (a) Show that $\{a_n\}$ converges to a in mean if $\{a_n\}$ converges to a.
   (b) Give a divergent sequence which converges in mean.

5. Let D_n be the Dirichlet kernel and define the Fejer kernel to be $F_n(x) = \dfrac{1}{n+1} \sum_{k=0}^{n} D_k(x)$.
   (a) Show that
$$F_n(x) = \frac{1}{2\pi(n+1)} \left(\frac{\sin(\frac{n+1}{2})x}{\sin x/2} \right)^2 \ , \quad x \ne 0 \ .$$
   (b) Let
$$\sigma_n f(x) = \frac{1}{n+1} \sum_{k=0}^{n} S_k f(x) \ .$$
   Show that for every $x \in [-\pi, \pi]$, $\sigma_n f(x)$ converges uniformly to $f(x)$ for any continuous, 2π-periodic function f. Hint: Follow the proof of Theorem 1.5 and use the non-negativity of F_n.

6. Let f and g be two continuous, 2π-periodic functions whose Fourier series are the same. Prove that $f \equiv g$. Hint: Show that $\int_{-\pi}^{\pi} fg \, dx = 0$ for all finite trigo series and apply Weierstrass approximation theorem.

7. Let $f, g \in R_{2\pi}$. Show that $\int_{-\pi}^{\pi} (f - g)^2(x) \, dx = 0$ and conclude that f and g are equal almost everywhere.